An intriguing problem from QCD

(suggested by E.O.)

Several years ago Pino Marchesini, while we were both at CERN to spend the Summer 2003, suggested that I should try to solve an equation he had just formulated with Al Mueller in the context of quark-antiquark scattering in perturbative QCD. The equation did not look too hard, having some experience with singular integral equations, but it had the nasty feature of a non-analytic kernel. Here it is

$$\partial_\tau \mathcal{I}(\rho,\tau) = \int_0^1 \frac{d\eta}{1-\eta}\left(\eta^{-1}\mathcal{I}(\eta\,\rho,\tau) - \mathcal{I}(\rho,\tau)\right) + \int_\rho^1 \frac{d\eta}{1-\eta}\left(\mathcal{I}(\rho/\eta,\tau) - \mathcal{I}(\rho,\tau)\right)$$

Nice, isn't it? Another equation was already around since a couple of years, known under the acronym **BFKL** (*Balitski-Fadin-Kuraev-Lipatov*), differing in a subtle detail which consists in the integration interval of the second integral being (0,1). A simple transformation brings the equation to a more transparent form. Let $\mathcal{I}(\rho,\tau) \equiv \rho\,\phi(\rho,\tau)$ and perform some simple transformation to get the following:

$$\partial_\tau \phi(\xi,\tau) = \int_0^1 \frac{\phi(\eta,\tau) - \phi(\xi,\tau)}{|\eta - \xi|}\, d\eta - \log\xi \cdot \phi(\xi,\tau)$$

If we consider the right hand side as an operator in $L_2([0,1])$, its first property is that it is an unbounded symmetric operator, possibly self-adjoint. The part involving a singular integral is easily recognised to have a very simple discrete spectrum. Actually, applying the integral to a power we get *($n > 0$)*

$$\int_0^1 \frac{\eta^n - \xi^n}{|\eta-\xi|}\, d\eta = -\int_0^\xi \sum_0^{n-1} \xi^k\,\eta^{n-1-k} d\eta + \int_\xi^1 \sum_0^{n-1} \xi^k\,\eta^{n-1-k} d\eta = -\,2\xi^n \sum_1^n k^{-1} + \mathcal{O}(\xi^{n-1})$$

which tells us that the integral operator leaves the subspace of polynomials of any degree n invariant and its spectrum is given by (-) twice the *harmonic numbers* $\sum_1^n 1/k$.

The "potential" $-\log \xi$ has to be taken into account, and this is the hard part of the problem. NOW: **the first challenge is to devise an efficient numerical algorithm to compute the spectrum of the full operator.** In Ref.[3] you may find some **hint.**

A more general form of the operator turns out to be already known in the literature, known as Tuck's equation, arising in hydrodynamics in shallow water. The references can be found in Ref.[4]. While an analytic solution has been found for the Marchesini-Mueller equation, mainly thanks to the genius of V. Fateev, for the original Tuck's equation no exact solution was known at the time (perhaps in the meanwhile…). The general form of the operator which requires an analytic solution is given by

$$K_{\alpha,\beta}\,\phi(\xi,\tau) = \int_0^1 \frac{\phi(\eta,\tau)-\phi(\xi,\tau)}{|\eta-\xi|}\,d\eta - (1-\alpha)\,\log\xi\cdot\phi(\xi,\tau) - (1-\beta)\,\log(1-\xi))\cdot\phi(\xi,\tau)$$

In Ref[4] you may find the exact spectrum of $K_{0,1}$. The **second challenge** is to find the spectrum for the general operator, either numerically or analytically.

References

1) G.Marchesini and A.H.Mueller, *BFKL dynamics in Jet evolution,* Phys.Lett. B 575 (2003) 37.

2) E. Tuck, J. Fluid Mech. 18(4) 619 (1964).

3) E. Onofri, *BFKL, MM, Alpert-Rokhlin transform, etc,* arXiv: physics/0407098v1 (unpublished).

4) V.A. Fateev et al. 2004 J.Phys. A: Math. Gen. 37, 11379.